INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 39, Northern Spring 2018 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Thirty nine non-zero numbers are written in a row. The sum of any two neighbouring numbers is positive, while the sum of all the numbers is negative. Is the product of all these numbers negative or positive? (4 points)

2. Aladdin has several gold coins and from time to time he asks the Genie to give him more. On each such occasion the Genie first responds by adding a thousand gold coins and then he takes back a half of the total weight of all Aladdin's gold coins. If after asking the Genie for more gold ten times, is it possible for Aladdin that the number of his gold coins has increased taking into account that each time the Genie takes a half of all Aladdin's gold back and no coin is broken into smaller pieces? (5 points)

3. Do there exist 2018 positive reduced fractions, each with a different denominator, such that the denominator of the difference of any two (after reducing to lowest terms) is less than the denominator of any of the initial 2018 fractions? (6 points)

4. Let O be the circumcentre of triangle ABC. Let AH be an altitude of triangle ABC, and let P be the foot of the perpendicular dropped to the line CO from point A. Prove that the line HP passes through the midpoint of the side AB. (6 points)

5. There are 100 houses in a street, which are divided into 50 pairs. Each pair are located opposite one another in the street. On the right side of the street all houses have even numbers, while all houses on the left side have odd numbers. On both sides of the street the numbers increase from one end of the street to the other, but the numbers are not necessarily consecutive (some numbers may be skipped). For each house on the right side of the street, the difference between its number and the number of the opposite house is calculated, and as it turns out, all the differences are distinct from one another. Let n be the greatest number of a house on the street. Find the smallest possible value of n. (8 points)

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6. In the land of knights (who always tell the truth) and knaves (who always lie), 10 people, at least one of them a knave, sit at a round table, each at a vertex of an inscribed regular 10-gon. A traveller can choose to stand at any point outside the table and ask the people at the table:

"What is the distance from me to the nearest knave at the table?"

After that each person at the table gives him an answer. What is the minimal number of questions the traveler has to ask to determine for sure what people at the table are knaves? (The people at the table and the traveller are to be considered as points, and everyone, including the traveller, can make exact measurement of the distance between any two points.) (10 points)

7. You are travelling to some country and you don't know its language. You know that symbols "!" and "?" stand for addition and subtraction, but you don't know which symbol is for which operation. Each of these two symbols can be written between two arguments, but for subtraction you don't know if the left argument is subtracted from the right or vice versa. For example, $a?b$ could mean any of $a-b$, $b-a$ and $a+b$. You don't know how to write any numbers, but variables and brackets can be used as usual. Given two arguments a and b how can you write for sure an expression that is equal to $20a-18b$? (12 points)

A Level Junior Paper Solutions

Edited by Oleksiy Yevdokimov and Greg Gamble

1. **Solution.** We show that the product of all the numbers is necessarily positive. First, consider any number having an odd-numbered position in the row, numbering the positions left to right: $1, 2, \dots, 39$. Then, all the remaining numbers can be divided into 19 pairs of neighbouring numbers whose respective sums are positive. Since the sum of all the numbers is negative, the number in the odd position under consideration must therefore be negative. Thus, all numbers in odd-numbered positions in the row must be negative. Now, since the sum of any two neighbouring numbers is positive, each number in an even-numbered position must be positive. Thus, there must be 20 negative numbers and 19 positive numbers in the row, and hence the product of all the numbers is positive.

   Note. Our argument above assumes the conditions can be satisfied. Examples satisfying the conditions do indeed exist, e.g. -20, 21, -20, $\dots$, 21, -20; in this example, each neighbouring pair sums to 1 (positive), but the total of all the numbers is -1 (negative).

2. **Solution 1.** No, it is not possible. Suppose Aladdin initially has $1000 + x$ gold coins. Then, after asking the Genie for more gold coins, once, Aladdin will have $1000 + x/2$ gold coins, and after asking the Genie for more coins, ten times, he will have $1000 + x/2^{10}$ gold coins. Since no coin is broken into smaller pieces, x must be divisible by 1024. Since Aladdin initially had a positive number of coins, $x > -1000$. Thus, for divisibility by 1024, x must in fact be non-negative, so that $1000 + x/2^{10} \le 1000 + x$, and hence, it is not possible that the number of Aladdin's gold coins could have increased.

   Solution 2. No, it is not possible. Going backwards, if Aladdin has x gold coins after asking the Genie once, then he had $2x - 1000$ gold coins before asking the Genie. Similarly, if Aladdin has x gold coins after asking the Genie ten times, then he had

   $$\begin{aligned} 2(2(\cdots(2x-1000)\cdots)-1000)-1000 &= 2^{10}x - 1000(2^9 + \cdots + 2 + 1) \\ &= 2^{10}x - 1000(2^{10}-1) \end{aligned}$$

   gold coins, initially.

   Suppose that the number of Aladdin's gold coins has increased. Then

   $$\begin{aligned} 2^{10}x - 1000(2^{10}-1) &< x \\ (2^{10}-1)x &< 1000(2^{10}-1) \\ x &< 1000 \end{aligned}$$

   However, $2^{10}x - 1000(2^{10}-1) > 0$, and so we get

   $$x > \frac{1000(2^{10}-1)}{2^{10}} = 1000\left(1 - \frac{1}{1024}\right) > 999$$

   which leads to a contradiction since we have $999 < x < 1000$ for an integer x.

3. **Solution 1.** Such 2018 positive reduced fractions do exist. Consider the fractions,

$$\frac{1+q}{q}, \frac{2+q}{2q}, \ldots, \frac{2018+q}{2018q},$$

where $q = 2018! + 1$. These fractions cannot be reduced since $(q, i) = 1$ for $1 \le i \le 2018$. The difference, of any two of the fractions above, can be written as

$$\frac{i+q}{iq} - \frac{j+q}{jq} = \frac{j(i+q) - i(j+q)}{ijq} = \frac{j-i}{ij},$$

where $1 \le i, j \le 2018$. Thus, the denominator of the difference of any two of the fractions is less than q, and hence less than q after reducing to lowest terms. So we are done.

Solution 2. Such 2018 positive reduced fractions do exist. Choose any 2018 positive reduced to lowest terms fractions with numerators $a_1, a_2, \ldots, a_{2018}$ and respective denominators $b_1 > b_2 > \cdots > b_{2018} > 0$. Choose a positive fraction of the form $1/d$ where $d > b_1 b_2$ and $(d, b_1 b_2 \cdots b_{2018}) = 1$. Then add $1/d$ to each of the 2018 chosen fractions, to obtain

$$\frac{a_i}{b_i} + \frac{1}{d_i} = \frac{a_i d + b_i}{b_i d},$$

for each i such that $1 \le i \le 2018$. The 2018 fractions thus obtained satisfy all requirements, since their reduced form denominators are db_i, as $(a_i d + b_i, db_i) = 1$, and the difference of any two of them,

$$\frac{a_i d + b_i}{b_i d} - \frac{a_j d + b_j}{b_j d} = \frac{(a_i d + b_i)b_j - (a_j d + b_j)b_i}{b_i b_j d} = \frac{a_i b_j - a_j b_i}{b_i b_j},$$

has denominator at most $b_1 b_2 < d < db_i$.

Solution 3 (by William Steinberg). Such 2018 positive reduced fractions do exist. Take 2019 primes $p_1 < p_2 < \cdots < p_{2018} < p_{2019}$. They are coprime; so, for each $i < 2019$, there exists b_i such that $b_i p_i \equiv 1 \pmod{p_{2019}}$.

By the Chinese Remainder Theorem, for each $i < 2019$, there exists an a_i satisfying the system of 2019 congruences,

$$\begin{aligned} a_i &\equiv b_i \pmod{p_{2019}} \\ a_i &\equiv 1 \pmod{p_j}, \text{ for } 1 \le j \le 2018. \end{aligned}$$

We will show that the 2018 fractions,

$$\frac{a_i p_i}{p_1 p_2 \cdots p_{2018} p_{2019}},$$

where $1 \le i \le 2018$, satisfy the requirements. Since a_i by design is coprime to each of the primes $p_1, p_2, \ldots, p_{2018}$, the reduced denominator of the ith fraction is

$$\frac{p_1 p_2 \cdots p_{2018} p_{2019}}{p_i}.$$

Now, for $1 \leq i, j \leq 2018$,

$$a_i p_i \equiv 1 \equiv a_j p_j \pmod{p_{2019}}.$$

So p_{2019} divides $a_i p_i - a_j p_j$. Hence the difference of the ith and jth fractions, $i \neq j$, when reduced to lowest terms, is at most

$$p_1 p_2 \cdots p_{2018} = \frac{p_1 p_2 \cdots p_{2018} p_{2019}}{p_{2019}} < \frac{p_1 p_2 \cdots p_{2018} p_{2019}}{p_i},$$

for $1 \leq i \leq 2018$.

4. **Solution.** Let M be the point of intersection HP and AB. We must show that M is the midpoint of AB. Indeed, since O is the circumcentre of triangle ABC,

$$\begin{aligned} \angle ABC &= \tfrac{1}{2}\angle AOC \\ &= 90^\circ - \angle OCA \\ &= \angle PAC. \end{aligned}$$

Since we have right angles at P and H standing on AC, $PACH$ is cyclic. Hence,

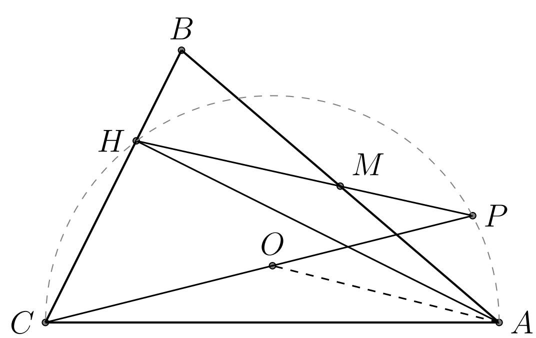
$$\angle PAC = 180^\circ - \angle PHC = \angle PHB.$$

Thus, triangle BMH is isosceles with $HM = BM$.

Let base angles at B and H of triangle AMH be α. Then, being exterior to triangle AMH, $\angle HMA = 2\alpha$. Also, $\angle MHA = 90^\circ - \alpha$. So,

$$\begin{aligned} \angle MAH &= 180^\circ - \angle HMA - \angle MHA \\ &= 180^\circ - 2\alpha - (90^\circ - \alpha) \\ &= 90^\circ - \alpha \\ &= \angle MHA. \end{aligned}$$

Therefore, triangle AMH is also isosceles, with $HM = AM$. Hence, $AM = BM$ which means M is the midpoint of AB.



5. **Solution 1.** The smallest possible value of n is 197. Recognising the problem as a discrete optimisation, we first find a minimum bound for n, and then demonstrate the bound is attainable via an example.

To estimate the difference between the greatest number of a house and the smallest number of a house on the street we need to consider the distribution of house numbers on both sides of the street bearing in mind that all the differences between a number of a house on the right side of the street and the number of the opposite house respectively are distinct from one another and odd. Therefore, there exist two differences that distinct from one another at least by 98. Note that house numbers along one side increase at least by 2. Thus, the difference between the greatest number of a house and the smallest number of a house on the street is at least $98 + 2 \times 49 = 196$ which gives the greatest number of a house on the street to be at least $1 + 196 = 197$. Consider the following example where houses on the right side have numbers $2, 4, 6, \ldots, 100$ and houses on the left side have numbers $1, 5, 9, \ldots, 197$, respectively. This example satisfies the statement of the problem. So the smallest possible value of n is 197.

Solution 2. The smallest possible value of n is 197. We again use bounding and demonstration by example that the bound is attainable. Let $a_1 < a_2 < \cdots < a_{50}$ be odd numbers of houses on the left side of the street and $b_1 < b_2 < \cdots < b_{50}$ be even numbers of houses on the right side of the street with $d_k = b_k - a_k$. Note that all d_k are odd. Let the least difference $d_i = d$ for some i, $1 \leq i \leq 50$ and the greatest difference $d_j = D$ for some j, $1 \leq j \neq i \leq 50$. Then, $D \geq d + 2 \times 49$.

Since we are looking for the smallest possible value of n, we can assume without loss of generality that $i > j$. Then,

$$a_i = b_i - d \geq b_j + 2(i - j) - d$$

and

$$a_i - a_j \geq D - d + 2(j - i) \geq 2(j - i) + 2 \times 49.$$

Therefore, $a_{50} - a_1 \geq 2 \times 49 + 2 \times 49 = 196$ with $a_{50} \geq 197$ follows. Using the same example as in Solution 1 we obtain the smallest possible value $n = 197$.

Note. The question does not state that all differences must be positive. If this condition holds, then an estimation similar to the estimation in Solution 2, but for the case $i < j$ gives the greatest number of a house on the street to be at least 198. Then such estimation together with the example where houses on the right side have numbers $2, 6, \ldots, 198$ and houses on the left side have numbers $1, 3, \ldots, 99$ respectively gives the smallest possible value of n to be 198.

Also note that the term *difference* conventionally is non-negative, though in the context of arithmetic sequences, a *common difference* is allowed to be negative. Thus usage of the word is ambiguous enough to allow either answer, if sufficiently clear which convention was taken.

6. **Solution.** The traveller has to ask two questions to determine for sure all knaves at the table. The first question can be asked from an arbitrary point. If all answers received are the same, then all people at the table are knaves since knight and knave give different answers. Otherwise, there exist neighbours at the table who give different answers. Then, the traveller must stay at the midpoint of the arc (which is part of the round table) between these two neighbours and ask the second question. Since at least one of these two neighbours is a knave, the distance from

the traveller to the nearest knave is known. Thus, those who respond with the distance correctly are knights and others are knaves.

Note that one question is insufficient. To see this, suppose the traveller asks just the one question. Assign all people at the table in groups according to distance from the traveller. Necessarily, the number of groups is at least two, since the traveller cannot be at the centre of the regular 10-gon, so the distances from him to the people at the table are not all the same. If the nearest group tell their distance to the next group and all other groups tell their distance to the nearest group, then the nearest group could be knights and all others could be knaves. However, the nearest group could be knaves and all others could be knights, and with one question we can't decide between these possibilities.

7. **Solution.** To write for sure any linear combination of a and b we need to know how we can represent 0, how we can represent the sum of two symbols a and b, and how we can represent the opposite symbol $-a$.

An expression $(a?a)!(a?a)$ is always equal to 0. So we can write 0 now bearing in mind that we mean $(a?a)!(a?a)$.

An expression $(a?0)?(0?b)$ is equal to $a+b$. Similarly to above, we can write $a+b$, bearing in mind that we mean $(a?0)?(0?b)$.

Furthermore, $0?((0!(a!0))?0)$ is always equal to $-a$. Thus, we can represent an expression that is equal to $20a-18b$ using the operations we have defined above:

$$\underbrace{((\cdots(a+a)+\cdots+a)+a)}_{\text{adding 20 symbols } a}+\underbrace{(-((\cdots(b+b)+\cdots+b)+b))}_{\text{adding 18 symbols } b}.$$

Note. The representations used for 0, $a+b$ and $-a$ are not unique. Other representations can be obtained by replacing "?" with "!" and vice versa.